Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, the given polar equation can be written in x and y as follows:

$$by^2 + 2xy - bx^2 \quad = \quad a. \tag{1}$$

Noting that (1) has the form $Dx^2 + Exy + Fy^2 = a$, the angle of rotation is found to be

$$\tan(2\theta) \quad = \quad \frac{E}{D-F} = -\frac{1}{b}. \tag{2}$$

With some perseverance and the standard rotation formulas with $x = u\,cos(\theta) - v\,\sin(\theta)$ and $y = u\,\sin(\theta) + v\,\cos(\theta)$, (1) can be written as

$$\big(\sin(2\theta) - b\,\cos(2\theta)\big)u^2 + \big(b\,\cos(2\theta) - \sin(2\theta)\big)v^2 \quad = \quad a. \tag{3}$$

Thus, using (2), $\sin(2\theta) = \dfrac{1}{\sqrt{b^2+1}}$ and $\cos(2\theta) = -\dfrac{b}{\sqrt{b^2+1}}$. (3) can now be simplified and displayed in standard form of a conic as

$$\sqrt{b^2+1}\,u^2 - \sqrt{b^2+1}\,v^2 = a$$

$$\frac{u^2}{\dfrac{a}{\sqrt{b^2+1}}} - \frac{v^2}{\dfrac{a}{\sqrt{b^2+1}}} = 1. \tag{4}$$

If we consider A to be the distance from the center of the hyperbola to a vertex, B to be the distance from the center to an end of the conjugate axis, and C to be the distance from the center to a focus, then from (4), $A^2 = \dfrac{a}{\sqrt{b^2+1}}$, $B^2 = \dfrac{a}{\sqrt{b^2+1}}$, and

$$C^2 \quad = \quad A^2 + B^2 = \frac{2a}{\sqrt{b^2+1}}. \tag{5}$$

Using (5), eccentricity is defined to be $e = \dfrac{C}{A} = \sqrt{2}$. Thus, there do not exist nonzero constants a and b to yield a rational eccentricity.

Editor's comment: This problem appeared before in this column as problem 5304; mea culpa, once again.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray; Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Toshihiro Shimizu, Kawasaki, Japan, and the proposer.

- **5387:** *Proposed by Arkady Alt, San Jose, CA*

Let $D := \{(x,y) \mid x, y \in R_+,\ x \neq y \text{ and } x^y = y^x\}$.(Obviously $x \neq 1$ and $y \neq 1$).
Find $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$

Solution 1 by Henry Ricardo, New York Math Circle, NY

The power mean inequality gives us

$$M_{-1}(x,y) \;=\; \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;\leq\; M_0(x,y) \;=\; \sqrt{xy},$$

so that

$$\sup_{(x,y)\in D}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;\leq\; \sup_{(x,y)\in D}\sqrt{xy}.$$

Now it is well known that the general solution of the equation $x^y = y^x$ in the first quadrant is given parametrically by

$$x \;=\; \left(1+\frac{1}{u}\right)^{u}, \quad y \;=\; \left(1+\frac{1}{u}\right)^{u+1}, \; u > 0,$$

a form attributed to Christian Goldbach. This gives us

$$x\cdot y \;=\; \left(1+\frac{1}{u}\right)^{u}\cdot\left(1+\frac{1}{u}\right)^{u+1},$$

implying that

$$\sup_{(x,y)\in D}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \;=\; \lim_{u\to\infty}\sqrt{xy} \;=\; \sqrt{e\cdot e} \;=\; e.$$

Solution 2 by Toshihiro Shimizu, Kawasaki, Japan

It is well-known that for any positive integer n,

$$(x,y) = \left(\left(1+\frac{1}{n}\right)^{n}, \left(1+\frac{1}{n}\right)^{n+1}\right)$$

satisfies the equation $x^y = y^x$ and $x \neq y$. Letting $n \to \infty$, both x and y converges to e. Thus, the value $((x^{-1}+y^{-1})/2)^{-1}$ also converges to e.
Next, we show that for any real number satisfying $x^y = y^x, x \neq y$, the equation $((x^{-1}+y^{-1})/2)^{-1} \leq e$ holds. $x^y = y^x$ is equivalent to $\log x/x = \log y/y$. Since $\log x/x$ is negative and monotone decreasing for $x < 1$, and it's positive and monotone increasing for $1 \leq x \leq e$ and also it's positive and monotone decreasing on $e \leq x$, it is obvious that $1 < x, y$ and without loss of generality , we assume $y < e < x$. We write $x = 1/s$, $y = 1/t$. Then, $s < 1/e < t$ and $s\log s = t\log t$. The inequality $((x^{-1}+y^{-1})/2)^{-1} \leq e$ is equivalent to $1/e \leq (s+t)/2$.
Let $f(x) = x\log x$. Then, $f'(x) = 1+\log x, f''(x) = 1/x, f'''(x) = -x^{-2} < 0$ for $x > 0$. Thus, $f'(x)$ is concave and it follows that

$$\frac{f'(z)+f'(\frac{2}{e}-z)}{2} \leq f'(\frac{z+\frac{2}{e}-z}{2}) = f'(\frac{1}{e}) = 0$$

for any $z > 0$. Integrating from $z = s$ to $z = 1/e$, we get

$$\frac{f(1/e) - f(s) + f(\frac{2}{e} - s) - f(1/e)}{2} \leq 0,$$

or $f(2/e - s) \leq f(s) = f(t)$. Since, $f(z)$ is monotone increasing on $1/e \leq z$, it follows that $2/e - s \leq t$ or $1/e \leq (s+t)/2$. Therefore we have shown that $((x^{-1} + y^{-1})/2)^{-1} \leq e$ for any $(x, y) \in D$.

Finally we conclude that the supremum value is e.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

In is known that $D \cap \left\{ (x,y) \middle| x \neq 1, y \neq 1 \right\}$ can be parametrized by

$(0,1) \cup (1, +\infty) \ni t \to \big(x(t), y(t)\ \big) = \left(t^{\frac{1}{t-1}}, t^{\frac{t}{t-1}}\right)$.

(Note that $t = \dfrac{y(t)}{x(t)}$ is the slope of the line from $(0,0)$ to $\big(x(t), y(t)\big)$; moreover,

$$y(t)^{x(t)} = \left(t^{\frac{t}{t-1}}\right)^{t^{\frac{1}{t-1}}} = t^{\frac{t}{t-1} \cdot t^{\frac{1}{t-1}}} = t^{\frac{t \cdot t^{\frac{1}{t-1}}}{t-1}} = t^{\frac{t^{1+\frac{1}{t-1}}}{t-1}} = t^{\frac{t^{\frac{t}{t-1}}}{t-1}} = t^{\frac{1}{t-1} \cdot t^{\frac{t}{t-1}}} = \left(t^{\frac{1}{t-1}}\right)^{t^{\frac{t}{t-1}}} = x(t)^{y(t)}.$$

Hence,

$$\left(\frac{x(t)^{-1} + y(t)^{-1}}{2}\right)^{-1} = \frac{2x(t)y(t)}{x(t) + y(t)} = \frac{2t^{\frac{1}{t-1}} \cdot t^{\frac{t}{t-1}}}{t^{\frac{1}{t-1}} + t^{\frac{t}{t-1}}} = \frac{2t^{\frac{1+t}{t-1}}}{t^{\frac{1}{t-1}} \cdot (1+t)} = \frac{2t^{\frac{t}{t-1}}}{t+1}.$$

Let us define $(0,1) \cup (1, \infty) \ni \mu \to f(u) = \dfrac{2u^{\frac{u}{u-1}}}{u+1.}$

Then $f'(u) = \dfrac{2u^{\frac{u}{u-1}} \left(2u - 2 - (u+1)\ln u\right)}{\left(u^2 - 1\right)^2}$ so $f'(u) > 0$ for $u \in (0,1)$ and $f'(u) < 0$ for $u \in (1, +\infty)$, with implies that f is strictly increasing in $(0,1)$ and strictly decreasing in $(1, +\infty)$, which implies that

$$\begin{aligned}
\sup_{u \in (0,1) \cup (1,+\infty)} f(u) &= \lim_{u \to 1} f(u) = \lim_{n \to 1} \frac{2}{u+1} \cdot \lim_{u \to 1} u^{\frac{u}{u-1}} = \lim_{n \to 1} u^{\frac{u}{u-1}} = e^{\ln \lim_{u \to 1} u^{\frac{u}{u-1}}} \\
&= e^{\lim_{u \to 1} \ln u^{\frac{u}{u-1}}} = e^{\lim_{u \to 1} \frac{u}{u-1} \ln u} = e^{\lim_{u \to 1} \frac{u}{u-1} \left(-\sum_{n=1}^{\infty} \frac{1-u^n}{n}\right)} = e^{\lim_{u \to 1} u \sum_{n=1}^{\infty} \frac{(1-u)^{n-1}}{n}} \\
&= e^{\lim_{u \to 1} u + \sum_{n=2}^{\infty} \frac{u(1-u)^{n-1}}{n}} = e^{1+0} = e.
\end{aligned}$$

Thus, $\displaystyle\sup_{(x,y) \in D} \left(\frac{x^{-1} + y^{-1}}{2}\right)^{-1} = \sup_{t \in (0,1) \cup (1,+\infty)} \left(\frac{x(t)^{-1} + y(t)^{-1}}{2}\right)^{-1} = \sup_{t \in (0,1) \cup (1,+\infty)} f(t) = e.$

Solutions 4 and 5 by Michael Brozinsky, Central Islip, NY

For simplicity, we shall use $\dfrac{2xy}{x+y}$, which equals the given expression.
We shall also use the Lambert function $W(x)$ which is the inverse of $f(x) = x \cdot e^x$ (with the domain of $f(x)$ being $\{-1, \infty\,)$ so that $W(x)$ has domain $\left[-\dfrac{1}{e}, \infty\right)$ and

$$\begin{aligned} W\left(x \cdot e^x\right) &= x \; if \; x \geq -1, \text{and} \\ x &= W(x) \cdot e^{W(x)}, \; if \; x \geq \frac{1}{e} \qquad (*) \end{aligned}$$

From $y^x = x^y$ we have $\dfrac{\ln(x)}{x} = \dfrac{\ln(y)}{y}$ (Δ), and since $F(t) = \dfrac{\ln(t)}{t}$ is one to one and negative on $(0,1)$, one to one and positive on $(1,e)$ and one to one and positive on (e,∞) and since $x \neq y$, we can assume that $1 < y < e$ and $x > e$ so that in particular $\ln(y) > -1$ and from $(*)$, $W\left(-\ln(y) \cdot e^{-\ln(y)}\right) = -\ln(y)$ which we will encounter later when we obtain $(**)$ below.

From $y^x = x^y$ we have by raising both sides to the $\dfrac{1}{xy}$ power that $y^{\frac{1}{y}} = x^{\frac{1}{x}}$. The left hand side can be written as $\left(e^{\ln(y)}\right)^{\frac{1}{y}} = \left(e^{\ln(y)}\right)^{e^{-\ln(y)}} = e^{\ln(y)\cdot e^{-\ln(y)}}$ and so we have $e^{\ln(y)\cdot e^{-\ln(y)}} = x^{\frac{1}{x}}$. If we take natural logs of both sides of this equation and multiply both sides by -1 we have

$$-\ln(y) \cdot e^{-\ln(y)} = \frac{-\ln(x)}{x} \qquad (1).$$

Now $\dfrac{-\ln(x)}{x} > -\dfrac{1}{e}$ (since $\dfrac{\ln(x)}{x}$ has it s maximum of $\dfrac{1}{e}$) when $x = e$ and thus $W\left(-\dfrac{\ln(x)}{x}\right) > -1$ and so $1 + W\left(-\dfrac{\ln(x)}{x}\right) > 0$. (Note $W(u) \geq -1$ with equality only if $u = -\dfrac{1}{e}$).

Taking W of both sides of (1) and using $(*)$ we have from (1) that
$-\ln(y) = W\left(-\dfrac{\ln(x)}{x}\right)$ $\quad(**)$ and so

$$y = \frac{1}{e^{-\ln(y)}} = \frac{1}{e^{W\left(-\frac{\ln(x)}{x}\right)}} = using(*) \; \frac{W\left(-\frac{\ln(x)}{x}\right)}{-\frac{\ln(x)}{x}} = -\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)$$

The expression whose supremum we wish to find is thus

$$\frac{2xy}{x+y} = \frac{2x\left(-\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)\right)}{x + \left(-\frac{x}{\ln(x)} \cdot W\left(-\frac{\ln(x)}{x}\right)\right)} - \frac{2x^2 W\left(-\frac{\ln(x)}{x}\right)}{\ln(x) \cdot \left(x - \frac{xW\left(-\frac{\ln(x)}{x}\right)}{\ln(x)}\right)} \qquad (***)$$

Now differentiating the second equation in $(*)$ shows $W'(x) = \dfrac{1}{e^{W(x)} \cdot (W(x)+1)}$ and so differentiating $(***)$ gives, after simplification

$$\frac{2W\left(-\frac{\ln(x)}{x}\right)^2 \left(\ln(x) - W\left(-\frac{\ln(x)}{x}\right) - 2\right)}{\left(\ln(x) - W\left(-\frac{\ln(x)}{x}\right)\right)^2 \left(1 + W\left(-\frac{\ln(x)}{x}\right)\right)} = -\frac{2\ln(y)^2\left(\ln(x)+\ln(y)-2\right)}{\left(\ln(x)+\ln(y)\right)^2\left(1-\ln(y)\right)} \; using \; (**) \;\; (1).$$

Recall $1 - \ln(y) = 1 + W\left(\frac{\ln(x)}{x}\right) > 0$. The expression in (1) thus is positive when $\ln(x) + \ln(y) - 2 < 0$ and negative when $\ln(x) + \ln(y) - 2 > 0$. This last expression in $(***)$ increases if $xy < e^2$ and decreases when $xy > e^2$ and thus has maximum of e when $xy = e^2$ and so e is the desired supremum.

Solution 5

For simplicity, we shall use $\dfrac{2xy}{x+y}$, which equals the given expression. From $y^x = x^y$ we have $\dfrac{\ln(x)}{x} = \dfrac{\ln(y)}{y}$ (Δ), and since $F1(t) = \dfrac{\ln(t)}{t}$ is one to one and negative on $(0,1)$, one to one and positive on $(1,e)$ and one to one and positive on (e,∞) and since $x \neq y$, we can assume that $1 < x < e$ and $y > e$

Now since $y \cdot \ln(x) = x \cdot \ln(y)$, we have that $y \cdot \ln(x) - x = x \cdot (\ln(y) - 1) > 0 \quad (*)$. Since $\dfrac{d}{dx}\left(u(x)^{v(x)}\right) = u(x)^{v(x)} \cdot \left(\dfrac{v(x)}{u(x)}u'(x) + \ln\left(u(x)\right) \cdot v'(x)\right)$ we readily have from $y^x = x^y$ by implicit differentiation that $y' = \dfrac{y \cdot \ln(y) - \frac{y^2}{x}}{y \cdot \ln(x) - x}$ and since $\dfrac{d}{dx}\left(\dfrac{2xy}{x+y}\right) = \dfrac{2\left(x^2y' + y^2\right)}{(x+y)^2}$ we have by substitution that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2xy}{x+y}\right) &= \frac{2y\left(\ln(y)x^2 + \ln(x)y^2 - 2xy\right)}{\left(y\ln(x) - x\right)(x+y)} \text{ and factoring out } xy \\
&= \frac{2xy^2\left(\frac{\ln(y)}{y}x + \frac{\ln(x)}{x}y - 2\right)}{(y\ln(x)-x)(x+y)^2}, \text{ and since } x^y = y^x, \\
&= \frac{2xy^2\left(\frac{\ln(x^y)}{y} + \frac{\ln(y^x)}{x} - 2\right)}{(y\ln(x)-x)(x+y)^2} \\
&= \frac{2xy^2\left(\ln(x) + \ln(y) - 2\right.}{\left(y\ln(x) - x\right)(x+y)^2}.
\end{aligned}$$

The expression is thus positive (recall $y\ln(x) - x > 0$) when $\ln(x) + \ln(y) - 2 < 0$ and negative when $\ln(x) + \ln(y) - 2 > 0$. Thus $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1} + y^{-1}}{2}\right)^{-1}$ increases if $xy < e^2$ and decreases when $xy > e^2$ and so e is the desired supremum.

Editor′s comment: Michael Brozinsky also submitted two more solutions to this problem, each in the spirit of solutions the above.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5388: *Proposed by Jiglău Vasile, Arad, Romania*

Let $ABCD$ be a cyclic quadrilateral, R and r its exradius and inradius respectively, and a, b, c, d its side lengths (where a and c are opposite sides.) Prove that

$$\frac{R^2}{r^2} \geq \frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}.$$

Solution 1 by Toshihiro Shimizu, Kawasaki, Japan

Remark: We assume that $ABCD$ is inscribable (and thus $ABCD$ is bicentric) and excircle is circumcircle.

Let the circumcircle and incircle of $ABCD$ be Γ(with center O), Γ'(with center I), respectively. We fix Γ,Γ' and move A such that $ABCD$ has circumcircle Γ and incircle Γ'. The existence of such quadrilateral is assured by the Poncelet's closure theorem (see also https://en.wikipedia.org/wiki/Poncelet%27s_closure_theorem).

If Γ and Γ' are concentric, the quadrilateral is square and we can easy to check that $R = \sqrt{2}r$ and $\frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2} = 2$. Thus the equality holds. We assume that Γ and Γ' are not concentric.

As A vary, we only show the case when (r.h.s), that is $\frac{a^2c^2}{b^2d^2} + \frac{b^2d^2}{a^2c^2}$, is maximum. The value is maximum when $\frac{ac}{bd}$ is maximum. We calculate the maximum value.

Let P be the intersection of AC and BD. Let W, X, Y, Z be the tangency point of Γ' with AB, BC, CD, DA, respectively.

Then, we show the following lemma. The point P is a fixed point as A varies. Let E be the intersection of AB and CD. Let F be the intersection of BC and DA. Since the quadrilateral $ABCD$ is inscribable, AC, BD, ZX, WY are all concurrent at point P. (it can be shown by Brianchon's theorem and we omit) Then, ZX is the polar line of F with respect to Γ' and WY is the polar line of E with respect to Γ'. Thus, FE is the polar line of P (intersection of ZX and WY) with respect to Γ'. Moreover, E, P is on the polar line of F with respect to Γ and F, P is on the polar line of E with respect to Γ. (This fact is well known and I saw it in my Japanese book.) Therefore, EF and P are polar line and pole with respect to both Γ and Γ'. We will show that this situation only occurs when P is one of the particular two points. More precisely, since EF is polar line of P with respect to both